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Letter to the Editor

On free vibration of cross-ply laminates in cylindrical bending

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1. Introduction

Cylindrical bending problem is one of the simplest problems of laminated plates [1]. Pagano [2] first derived an exact static three-dimensional (3-D) solution of cross-ply laminates subject to cylindrical bending. He further presented an exact solution of angle-ply laminates [3]. Pagano's two exact solutions are only restricted to a laminate where edges are simply supported. Studies on other type boundary conditions basically based on various plate theories are carried out, and significant achievements have been made recently [4–6].

Differential quadrature method (DQM) has been proved to be very effective in solving differential equations governing beam/plate/shell deformations and vibrations [7–9]. On the other hand, the state-space method (SSM) is very effective in analyzing laminated structures [10–13]. However, exact solutions are available only for simply supported conditions. To overcome this difficulty, Chen et al. [14] recently developed a semi-analytical method combining DQM and SSM together and successfully analyzed the free vibration of sandwich beams. The method allows us to deal with different boundary conditions exactly that the Saint-Venant principle becomes unnecessary in the analysis. In this paper, the method is extended to analyze the free vibration of cross-ply laminates in cylindrical bending. The results presented here are believed to be valuable, especially those for thick laminates with non-simply supported conditions.

2. Basic elasticity formulations

As in Pagano [2], we consider an N-layered cross-ply laminate under the assumption of cylindrical bending (Fig. 1). In this case, we have only two non-zero displacements u and w, in x and z directions, respectively, which are independent of the co-ordinate y. The constitutive

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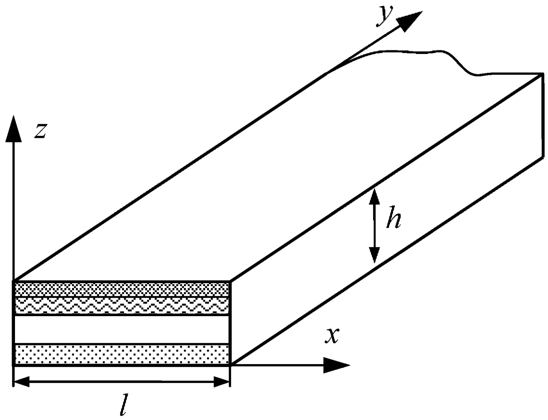

Fig. 1. A laminated plate in cylindrical bending.

relations for a cross-ply laminate for cylindrical bending are

$$\sigma_x = c_{11}\frac{\partial u}{\partial x} + c_{13}\frac{\partial w}{\partial z}, \quad \sigma_y = c_{12}\frac{\partial u}{\partial x} + c_{23}\frac{\partial w}{\partial z},$$
$$\sigma_z = c_{13}\frac{\partial u}{\partial x} + c_{33}\frac{\partial w}{\partial z}, \quad \tau_{xz} = c_{55}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \tag{1}$$

and $\tau_{xy} = \tau_{yz} = 0$, where c_{ij} are the elastic constants, and σ_i and τ_{ij} are the normal and shear stress components, respectively. The equations of motion are

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} = \rho\frac{\partial^2 u}{\partial t^2}, \quad \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \sigma_z}{\partial z} = \rho\frac{\partial^2 w}{\partial t^2}, \tag{2}$$

where ρ is the mass density.

Following the routine procedure of SSM [10,11], the following state equation can be established from Eqs. (1) and (2):

$$\frac{\partial}{\partial z}\begin{Bmatrix}\sigma_z\\ u\\ w\\ \tau_{xz}\end{Bmatrix} = \begin{bmatrix} 0 & 0 & \rho\frac{\partial^2}{\partial t^2} & -\frac{\partial}{\partial x}\\ 0 & 0 & -\frac{\partial}{\partial x} & \frac{1}{c_{55}}\\ \frac{1}{c_{33}} & -\frac{c_{13}}{c_{33}}\frac{\partial}{\partial x} & 0 & 0\\ -\frac{c_{13}}{c_{33}}\frac{\partial}{\partial x} & \rho\frac{\partial^2}{\partial t^2} + \alpha\frac{\partial^2}{\partial x^2} & 0 & 0 \end{bmatrix}\begin{Bmatrix}\sigma_z\\ u\\ w\\ \tau_{xz}\end{Bmatrix}, \tag{3}$$

where $\alpha = -c_{11} + c_{13}^2/c_{33} \cdot \sigma_z$, u, w and τ_{xz} are termed as basic variables, from which the two induced variables can be determined as

$$\sigma_x = \frac{c_{13}}{c_{33}}\sigma_z - \alpha\frac{\partial u}{\partial x}, \quad \sigma_y = \frac{c_{23}}{c_{33}}\sigma_z + \left(c_{12} - \frac{c_{23}c_{13}}{c_{33}}\right)\frac{\partial u}{\partial x}. \tag{4}$$

In the regime of 3-D elasticity, the typical boundary conditions can be expressed as

$$\begin{aligned} \text{simply supported (S)}: &\quad \sigma_x = w = 0, \\ \text{clamped (C)}: &\quad u = w = 0, \\ \text{free (F)}: &\quad \sigma_x = \tau_{xz} = 0, \end{aligned} \tag{5}$$

at $x = 0$ or l.

3. Exact solution

An exact solution for simply supported boundary conditions can be obtained, as shown by many researchers [1,2,11,15]. To show the solving procedure of SSM, we briefly review the analysis here with particular dimensionless notations. We assume

$$\begin{Bmatrix} \sigma_z \\ u \\ w \\ \tau_{xz} \end{Bmatrix} = \begin{Bmatrix} -c_{55}^{(1)}\bar{\sigma}_z(\zeta)\sin(n\pi\xi) \\ h\bar{u}(\zeta)\cos(n\pi\xi) \\ h\bar{w}(\zeta)\sin(n\pi\xi) \\ c_{55}^{(1)}\bar{\tau}_{xz}(\zeta)\cos(n\pi\xi) \end{Bmatrix} \exp(\mathrm{i}\omega t), \tag{6}$$

where ω is the circular frequency, $\zeta = z/h$ and $\xi = x/l$ are the dimensionless co-ordinates, n is an integer, and $c_{55}^{(1)}$ represents the elastic constant of the first layer (the bottom layer). It is readily shown that the simply supported conditions in Eq. (5) have been satisfied identically. The substitution of Eq. (6) into Eq. (3) yields

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}\mathbf{V}(\zeta) = \mathbf{A}\mathbf{V}(\zeta), \tag{7}$$

where $\mathbf{V}(\zeta) = [\bar{\sigma}_z(\zeta), \bar{u}(\zeta), \bar{w}(\zeta), \bar{\tau}_{xz}(\zeta)]^{\mathrm{T}}$, and the constant coefficient matrix $\mathbf{A}$ can be easily obtained, which is omitted here for brevity. The solution of $\mathbf{V}(\zeta)$ in Eq. (7) can be obtained as

$$\mathbf{V}(\zeta) = \exp[\mathbf{A}(\zeta - \zeta_{i-1})]\mathbf{V}(\zeta_{i-1}), \quad (\zeta_{i-1} \leqslant \zeta \leqslant \zeta_i,\ i = 1, 2, \ldots, N), \tag{8}$$

where $\zeta_0 = 0$, $\zeta_i = \sum_{j=1}^{i} h_j/h$, and h_i is the thickness of the ith layer. Because of the continuity conditions at each interface, we obtain from Eq. (8)

$$\mathbf{V}(1) = \mathbf{T}\mathbf{V}(0), \tag{9}$$

where $\mathbf{T} = \prod_{j=N}^{1} \exp[\mathbf{A}(\zeta_j - \zeta_{j-1})]$ is known as the global transfer matrix. After applying the tractions-free boundary conditions at the upper and lower surfaces, i.e., at $\zeta = 1$ and 0, respectively, one can derive the condition

$$\begin{vmatrix} T_{12} & T_{13} \\ T_{42} & T_{43} \end{vmatrix} = 0, \tag{10}$$

where T_{ij} are elements of the matrix $\mathbf{T}$, from which one can compute the natural frequency of the laminates.

It is interesting to mention that if we replace $\sin(n\pi\xi)$ by $\cos(n\pi\xi)$ and vice versa in Eq. (6), an exact solution can be similarly obtained. In this case, the edge boundary condition corresponds to the so-called guided [16] (or rigidly slipping or rigidly smooth contact) condition, i.e.,

$$\tau_{xz} = u = 0 \quad \text{at } x = 0, l. \tag{11}$$

4. The semi-analytical method

For a laminate with other boundary conditions, say clamped or free boundary conditions, it is generally difficult to obtain an exact solution to Eq. (3). A semi-analytical solution utilizing the principle of differential quadrature was recently proposed in Ref. [14]. In this method, the nth order partial derivative of any continuous function $f(x)$ at a given point x_i is approximated as a linear sum of weighted function values at all of the discrete points in the domain of x:

$$\left.\frac{\partial^n f(x)}{\partial x^n}\right|_{x=x_i} = \sum_{j=1}^{M} g_{ij}^{(n)} f(x_j) \quad (n = 1, 2, \ldots, M-1;\ i = 1, 2, \ldots, M), \tag{12}$$

where M is the number of discrete points, $g_{ij}^{(n)}$ are weight coefficients depending on $x_i\,(i = 1, 2, \ldots, M)$ only, and the corresponding expressions can be found in Ref. [17].

By virtue of the rule in Eq. (12), we obtain the following discrete forms of Eqs. (3) and (4), with respect to the variable ξ:

$$\begin{aligned}
\frac{\mathrm{d}Z_i}{\mathrm{d}\zeta} &= -\frac{\rho}{\rho^{(1)}}\Omega^2 W_i - s\sum_{j=1}^{M} g_{ij}^{(1)} T_j,\\
\frac{\mathrm{d}U_i}{\mathrm{d}\zeta} &= -s\sum_{j=1}^{M} g_{ij}^{(1)} W_j + \frac{c_{55}^{(1)}}{c_{55}} T_i,\\
\frac{\mathrm{d}W_i}{\mathrm{d}\zeta} &= \frac{c_{55}^{(1)}}{c_{33}} Z_i - \frac{c_{13}}{c_{33}} s\sum_{j=1}^{M} g_{ij}^{(1)} U_j,\\
\frac{\mathrm{d}T_i}{\mathrm{d}\zeta} &= \frac{\alpha}{c_{55}^{(1)}} s^2 \sum_{j=1}^{M} g_{ij}^{(2)} U_j - \frac{\rho}{\rho^{(1)}}\Omega^2 U_i - \frac{c_{13}}{c_{33}} s \sum_{j=1}^{M} g_{ij}^{(1)} Z_j,
\end{aligned} \tag{13}$$

$$X_i = \frac{c_{13}}{c_{33}} Z_i - \frac{\alpha}{c_{55}^{(1)}} s\sum_{j=1}^{M} g_{ij}^{(1)} U_j, \quad Y_i = \frac{c_{23}}{c_{33}} Z_i + \frac{c_{12}c_{33} - c_{13}c_{23}}{c_{33}c_{55}^{(1)}} s\sum_{j=1}^{M} g_{ij}^{(1)} U_j, \tag{14}$$

where $i = 1, 2, \ldots, M$, $\Omega = h\omega\sqrt{\rho^{(1)}/c_{55}^{(1)}}$, $\rho^{(1)}$ is the mass density of the first layer, $s = h/l$ is the thickness-to-span ratio, and

$$\begin{aligned}
&Z_i = \sigma_z(\zeta, \xi_i)/c_{55}^{(1)}, \quad U_i = u(\zeta, \xi_i)/h, \quad W_i = w(\zeta, \xi_i)/h,\\
&T_i = \tau_{xz}(\zeta, \xi_i)/c_{55}^{(1)}, \quad X_i = \sigma_x(\zeta, \xi_i)/c_{55}^{(1)}, \quad Y_i = \sigma_y(\zeta, \xi_i)/c_{55}^{(1)}
\end{aligned} \tag{15}$$

in which ξ_i are the sampling point co-ordinates determined by

$$\xi_i = \frac{1 - \cos[(i-1)\pi/(M-1)]}{2} \quad (i = 1, 2, \ldots, M). \tag{16}$$

These points are known as the Chebyshev–Gauss–Lobatto points [7].

At this stage, we have derived a discrete form of the state equation, as shown in Eq. (13). However, for a practical problem, one should take account of the edge boundary conditions before solving it. To illustrate the idea, we consider, for instance, a cantilever laminate clamped at $\xi = 0$ and free at $\xi = 1$:

$$U_1 = W_1 = X_M = T_M = 0. \tag{17}$$

From the third condition in Eq. (17) and the first equation in Eq. (14), we have

$$Z_M = \frac{c_{33}\alpha}{c_{13}c_{55}^{(1)}} s \sum_{j=1}^{M} g_{Mj}^{(1)} U_j. \tag{18}$$

Then we can derive from Eq. (13) the final form of discrete state equation:

$$\begin{aligned}
\frac{\mathrm{d}Z_i}{\mathrm{d}\zeta} &= -\frac{\rho}{\rho^{(1)}}\Omega^2 W_i - s\sum_{j=1}^{M-1} g_{ij}^{(1)} T_j \quad (i = 1, 2, \ldots, M-1),\\
\frac{\mathrm{d}U_i}{\mathrm{d}\zeta} &= -s\sum_{j=2}^{M} g_{ij}^{(1)} W_j + \frac{c_{55}^{(1)}}{c_{55}} T_i \quad (i = 2, 3, \ldots, M),\\
\frac{\mathrm{d}W_i}{\mathrm{d}\zeta} &= \frac{c_{55}^{(1)}}{c_{33}} Z_i - \frac{c_{13}}{c_{33}} s\sum_{j=2}^{M} g_{ij}^{(1)} U_j \quad (i = 2, 3, \ldots, M-1),\\
\frac{\mathrm{d}W_M}{\mathrm{d}\zeta} &= \left(\frac{\alpha}{c_{13}} - \frac{c_{13}}{c_{33}}\right) s\sum_{j=2}^{M} g_{Mj}^{(1)} U_j,
\end{aligned}$$

$$\begin{aligned}
\frac{\mathrm{d}T_i}{\mathrm{d}\zeta} =& \frac{\alpha}{c_{55}^{(1)}} s^2 \sum_{j=2}^{M} [g_{ij}^{(2)} - g_{iM}^{(1)} g_{Mj}^{(1)}]\, U_j\\
& - \frac{\rho}{\rho^{(1)}}\Omega^2 U_i - \frac{c_{13}}{c_{33}} s\sum_{j=1}^{M-1} g_{ij}^{(1)} Z_j \quad (i = 1, 2, \ldots, M-1),
\end{aligned} \tag{19}$$

which can be rewritten in the matrix form

$$\frac{\mathrm{d}\mathbf{V}_d}{\mathrm{d}\zeta} = \mathbf{A}_d \mathbf{V}_d, \tag{20}$$

where $\mathbf{V}_d = [Z_1, Z_2, \ldots, Z_{M-1}, U_2, U_3, \ldots, U_M, W_2, W_3, \ldots, W_M, T_1, T_2, \ldots, T_{M-1}]^{\mathrm{T}}$ is the discrete state vector, $\mathbf{A}_d$ is the coefficient matrix whose elements can be obtained from Eq. (19) easily. Similar to the derivation in the last section, we can arrive at

$$\mathbf{V}_d(1) = \mathbf{T}_d \mathbf{V}_d(0), \tag{21}$$

where $\mathbf{T}_d = \prod_{j=N}^{1} \exp[\mathbf{A}_d(\zeta_j - \zeta_{j-1})]$ is the global transfer matrix.

For free vibration problem, both the upper and lower surfaces are traction free; thus the frequency equation can be obtained from Eq. (21) as follows:

$$\begin{vmatrix} S_{1,M} & \cdots & \cdots & S_{1,3M-3} \\ \vdots & & & \vdots \\ S_{M-1,M} & \cdots & \cdots & S_{M-1,3M-3} \\ S_{3M-2,M} & \cdots & \cdots & S_{3M-2,3M-3} \\ \vdots & & & \vdots \\ S_{4M-4,M} & \cdots & \cdots & S_{4M-4,3\,M-3} \end{vmatrix} = 0, \tag{22}$$

where $S_{k,l}$ is the element of $\mathbf{T}_d$.

5. Numerical investigations

The convergence characteristics of the semi-analytical method described above are checked. The first example we consider is the free vibration of a simply supported (S-S) laminate, for which the exact frequencies are calculated from Eq. (10). The following typical material properties are adopted:

$$E_{\mathrm{L}}/E_{\mathrm{T}} = 25, \quad G_{\mathrm{LT}}/E_{\mathrm{T}} = 0.5, \quad G_{\mathrm{TT}}/E_{\mathrm{T}} = 0.2, \quad v_{\mathrm{LT}} = v_{\mathrm{TT}} = 0.25, \tag{23}$$

where E is Young's modulus, G the shear modulus, v the Poisson ratio and subscripts L and T indicate directions parallel and perpendicular to the fibers respectively. For all cases to be considered, each layer involved in the N-layered laminate is considered to have an equal thickness h/N. In this paper, the stacking sequence is always from the top ($\zeta = 1$) to bottom ($\zeta = 0$), and the notation system by Whitney [1] is adopted.

For a moderately thick laminate with $s = 0.1$, Table 1 compares the lowest dimensionless frequencies $\omega^* = \omega h\sqrt{\rho/G_{\mathrm{LT}}}$ obtained by different methods. Among them, D2D and M2D are the generalized displacement-based and generalized mixed-based plate theories, respectively, and the followed numbers indicate the terms associated with higher order displacements or shear stresses involved in these two plate theories [6]. The 3-D results are also directly cited from Ref. [6], calculated by the transfer matrix method [15] that is similar to that described in Section 3 of this paper. It is found that all the results calculated from Eq. (10) are identical to Messina's 3-D solutions [6], except for the stacking sequence $[(0/90)_3 0^{\circ}_1]_5$. However, our semi-analytical results always agree well with that obtained from Eq. (10). In particular, when the number of sampling points is taken as $M = 7$, the results are almost identical with the exact results. In Table 1 and hereafter, the relative error is defined as follows:

$$e\% = \frac{(\omega^* - \omega_0^*)}{\omega_0^*} \times 100, \tag{24}$$

where ω_0^* corresponds to the frequency parameter calculated from Eq. (10). Even for very thick laminate with $s = 0.3$ and 0.4, the semi-analytical method provides a very good estimate of the lowest frequencies, as shown in Table 2.

Table 1
Lowest frequency parameters $\omega^* = \omega h\sqrt{\rho/G_{LT}}$, for different stacking sequences of S-S laminate in cylindrical bending ($s = 0.1$)

Method	[0/90°]	e%	[0/90/0°]	e%	[0/90/0/90°]	e%
D2D, 8	0.0816460	−0.06	0.147297	0.72	0.110153	0.63
M2D, 11	0.0816126	−0.10	0.146203	−0.03	0.109414	−0.04
Semi, M						
5	0.0813411	−0.43	0.145721	−0.36	0.109036	−0.39
6	0.0816722	−0.03	0.146221	−0.02	0.109439	−0.02
7	0.0816958	0.00	0.146249	0.00	0.109462	0.00
Eq. (10)	0.0816952	–	0.146248	–	0.109461	–
3-D	0.0816952	–	0.146248	–	0.109461	–
	$[(0/90)_2 0°]$	e%	$[0/90°]_6$	e%	$[(0/90)_3 0^\circ_1]_5$	e%
D2D, 8	0.141136	0.89	0.122924	2.98	0.134013	
M2D, 11	0.139862	−0.02	0.119337	−0.03	0.129748	
Semi, M						
5	0.139372	−0.37	0.118902	−0.39	0.127260	−0.39
6	0.139865	−0.02	0.119344	−0.02	0.127731	−0.02
7	0.139892	0.00	0.119369	0.00	0.127757	0.00
Eq. (10)	0.139891	–	0.119368	–	0.127756	–
3-D	0.139891	–	0.119368	–	0.129758	

Table 2
Lowest frequency parameters $\omega^* = \omega h\sqrt{\rho/G_{LT}}$, for different stacking sequences of S-S very thick laminate in cylindrical bending

	[0/90°]	e%	[0/90/0°]	e%	[0/90/0/90°]	e%	$[(0/90)_2 0°]$	e%	$[0/90°]_6$	e%	$[(0/90)_3 0^\circ_1]_5$	e%
Semi, M ($s = 0.3$)												
5	0.523172	−0.34	0.646479	−0.28	0.541361	−0.30	0.627125	−0.28	0.579050	−0.28	0.614362	−0.28
6	0.524858	−0.02	0.648217	−0.01	0.542889	−0.01	0.628768	−0.01	0.580606	−0.01	0.616009	−0.01
Eq. (10)	0.524948	–	0.648310	–	0.542970	–	0.628856	–	0.580689	–	0.616097	–
Semi, M ($s = 0.4$)												
5	0.782482	−0.31	0.914114	−0.27	0.777475	−0.29	0.877888	−0.27	0.814024	−0.27	0.863234	−0.27
6	0.784787	−0.02	0.916508	−0.01	0.779594	−0.01	0.880118	−0.01	0.816095	−0.01	0.865464	−0.01
Eq. (10)	0.784910	–	0.916633	–	0.779708	–	0.880235	–	0.816199	–	0.865538	–

Table 3 presents the lowest frequency parameter ω^* of a cross-ply laminate with the stacking sequence [0/90/0/90°] for different wave numbers n. It should be noted that the wave number n is not involved in the formulations of the semi-analytical method. However, through the calculation of vibration modes, it is easy to determine the mode number (exact theory) the frequency (the semi-analytical method) corresponds to. It can be seen that, the higher the mode number is, the more is the number of sampling points required to get reasonable results. For example, as listed in Table 3, the frequency for $n = 6$ obtained by our method has a relative error of −1.76%

Table 3
Lowest frequency parameters, ω^*, for S-S laminate of [0/90/0/90°] ($s = 0.1$)

n	3-D	Eq. (10)	D2D,7	e%	M2D,7	e%	Semi, 12	e%
1	0.109461	0.109461	0.112068	0.7	0.109239	−0.2	0.109461	0.00
2	0.316561	0.316561	0.321023	1.4	0.315550	−0.3	0.316572	0.00
3	0.542971	0.542970	0.551860	1.6	0.541454	−0.3	0.542968	0.00
4	0.779708	0.779708	0.792924	1.7	0.778136	−0.2	0.779632	−0.01
5	1.02465	1.02465	1.04218	1.7	1.02333	−0.1	1.02682	0.21
6	1.27545	1.27545	1.29749	1.7	1.27461	−0.1	1.25305	−1.76

Table 4
Lowest frequency parameters, ω^*, for C-C laminate ($M = 8$)

s	[0/90°]	[0/90/0°]	[0/90/0/90°]	$[(0/90)_2 0°]$	$[0/90°]_6$	$[(0/90)_3 0^\circ_1]_5$
0.01	0.00197652 (0.00198100)	0.00439808 (0.00449054)	0.00298711 (0.00302247)	0.00402442 (0.00409065)	0.00323169 (0.00326793)	0.00348278 (0.00352398)
0.05	0.0458420 (0.0495249)	0.0791464 (0.112264)	0.0601057 (0.0755619)	0.0758616 (0.102266)	0.0653316 (0.0816982)	0.0698304 (0.0880995)
0.1	0.153462 (0.198100)	0.205606 (0.449054)	0.169593 (0.302247)	0.200086 (0.409065)	0.181942 (0.326793)	0.193064 (0.352398)
0.2	0.417144 (0.792398)	0.474495 (1.79622)	0.410782 (1.20899)	0.455344 (1.63626)	0.421828 (1.30717)	0.445170 (1.40959)

Table 5
Lowest frequency parameters, ω^*, for C-F laminate ($M = 8$)

s	[0/90°]	[0/90/0°]	[0/90/0/90°]	$[(0/90)_2 0°]$	$[0/90°]_6$	$[(0/90)_3 0^\circ_1]_5$
0.01	0.000311205 (0.000311322)	0.000704284 (0.00070571)	0.000474441 (0.000474995)	0.000641845 (0.000642863)	0.000513005 (0.000513569)	0.000553173 (0.000553809)
0.05	0.00771633 (0.00778305)	0.0168268 (0.0176427)	0.0115482 (0.0118749)	0.0154734 (0.0160716)	0.0125031 (0.0128392)	0.0134661 (0.0138452)
0.1	0.0301118 (0.0311322)	0.059896 (0.070571)	0.0429367 (0.0474995)	0.0561013 (0.0642863)	0.0465455 (0.0513569)	0.0499804 (0.0553809)
0.2	0.110421 (0.124529)	0.179206 (0.282283)	0.139518 (0.189998)	0.172043 (0.257145)	0.150410 (0.205428)	0.160387 (0.221524)

when compared to the exact one. However, if we take $M = 14$, the result will be 1.27431, with the error being 0.1% only.

Tables 4 and 5 show the lowest frequency parameters ω^* of C-C and C-F laminates respectively for several different thickness-to-span ratios. For comparison purpose, results calculated by the classical laminate theory (CLT [1]) are simultaneously given in parentheses. The results presented in these two tables calculated by our semi-analytical method are for $M = 8$. These results are accurate enough when compared to those for larger M. For example, for the stacking sequence [0/90/0/90°] and $s = 0.2$, the lowest frequency parameter ω^* of the C-C laminate equals 0.410732 for $M = 9$, differing very slightly from that for $M = 8$, as listed in Table 4.

From Tables 4 and 5, it is seen that for a cantilever laminate, the CLT results agree well with the semi-analytical results even for $s = 0.05$. However, for the C-C laminate, it is valid only for $s = 0.01$. Also, for different stacking sequences, the results of CLT deviate considerably from those of the present method.

6. Conclusion

The free vibration of a cross-ply laminate in cylindrical bending is studied. In contrast to the plate theories in which deformations or stresses are generally approximated in the thickness direction, we assume an approximation in the plate plane using the differential quadrature method. The present method overcomes the difficulty encountered in the traditional SSM when treating with non-simply supported boundary conditions, such as the clamped and free-end conditions. Moreover, since it is exactly solved along the thickness direction, the method is suitable for analyzing arbitrarily thick laminates. Various numerical results, especially the ones for non-simply supported thick laminates, are provided. And the authors hope them to serve as benchmark solutions for verifying approximate theories.

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